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Controller Design for Polynomial Discrete-Time Systems

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***Abstract*— In this paper we propose a nonlinear optimal controller design for polynomial discrete-time systems. Notations, properties of the tensor algebra and the polynomial description are used to derive analytically a nonlinear polynomial controller. Finally, a numerical example is provided to illustrate the validity of the proposed method**

Keywords— polynomial discrete-time systems, optimal control, Kronecker product, quadratic performance index.

I. INTRODUCTION

The last years, the optimal control of nonlinear systems is considered to be one of the most challenging control problems that have attract the attention of many researchers. Recently, an increasing amount of work has been devoted to the study of polynomial systems when the nonlinear systems are expressed in polynomial vector fields [1],[2],[3],[21] and various approaches are cited in the literature such as the traditional infinite-horizon optimal control [4],[17] addressed for continuous and discrete-time systems [16], [14], [18], [19], the minimax optimal control [5],[6],[7],[16] and the online optimal control [9],[11]. Besides, many others techniques investigate the optimal control of specific classes of systems, but it remains much to be done especially with the nonlinear discrete systems.

We have to note that the optimal control of nonlinear discrete time systems often requires solving either nonlinear partial difference or differential Hamilton-Jacobi-Bellman (HJB) equation which is a difficult task and need important algebraic manipulations. Therefore to overcome these difficulties we introduce the notations, properties and algebraic laws of the Kronecker product [22],[1],[2]. The control law is chosen to have the same polynomial structure of the state description of the studied system. the main contribution of the proposed optimization technique is to reduce the resolution of the initial problem to the investigation of an augmented Riccati equation instead of the resolution of the HJB equation or the State Dependent Riccati Equation SDRE equation.

The remainder of this paper is organized as follows. In section 2, the studied system and the problem statement are presented. In section 3, the new technique of nonlinear polynomial system is derived by the determination of the gain matrices and the control law is determined, followed by section 4 where the proposed method is evaluated by a numerical example. Finally a few concluding remarks are provided.

II. PROBLEM STATEMENT

In this work, we focus on polynomial discrete systems described by the following state-space equation:

$$X_{k+1} = A_1 X_k + A_2 X_k^{[2]} + A_3 X_k^{[3]} + BU_k \qquad (1)$$

where k is the discrete time index, $X_k^{[i]} \in \mathcal{R}^{n^i}$ is the i-th Kronecker power of the state vector, $U_k \in \mathcal{R}^m$, is the input vector, $A_i(n \times n^i)$ and $B(n \times m)$ are constant matrices describing the system parameter.

To control (1) in an optimal manner, it is required to select the control that minimizes the infinite horizon cost:

$$J = \sum_{k=0}^{\infty} (X_k^T Q X_k + U_k^T R U_k) \qquad (2)$$

where Q and R are symmetric positive-definite weighting matrices. This control law have to ensure that the system under control is asymptotically stable. The main contribution of this paper is that the proposed nonlinear law can be obtained via solving an optimization problem in terms of an augmented Riccati equation , instead of state-dependent Riccati equation

(SDRE) or Hamilton–Jacobi equations that are usually required in solving nonlinear optimal control problems.

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III STATE OPTIMAL CONTROLLER DESIGN

For the polynomial nonlinear discrete-time system described in (1), a state feedback controller is performed as:

$$U_k = -(K_1 X_k + K_2 X_k^{[2]} + K_3 X_k^{[3]}) \tag{3}$$

where K_i , i = 1,...,3 are constant gain matrices.

The system (1) provided by the polynomial control law (3) can be written as follows:

$$X_{k+1} = (A_1 - BK_1)X_k + (A_2 - BK_2)X_k^{[2]} + (A_3 - BK_2)X_k^{[3]} \tag{4}$$

where $A_i - BK_i$, i = 1,...,3 are the closed loop system matrices and K_i , i = 1,...,3 are to be determined according to the following proposed strategy based on two steps.

A. *First step : determination of* $\boldsymbol{K_1}$

By linearizing the system on the near neighborhood of the origin ($X = 0$), we obtain a discrete-time linear dynamical system and the equation of the closed loop system can be written as following :

$$X_{k+1} = (A_1 - BK_1)X_k \tag{5}$$

the control law is also linear and can be approximated by the following equation :

$$U_k = -K_1 X_k \tag{6}$$

K_1 is the gain which minimize the quadratic cost J, is gived by the following expression :

$$K_1 = (R + B^T PB)^{-1} B^T PA_1 \tag{7}$$

Where P is the positive-definite matrix, the solution of the Ricatti discrete-equation and it is described by the following equation :

$$P = (A_1 - BK_1)^T P (A_1 - BK_1) + K_1^T RK_1 + Q \tag{8}$$

B. *Second step : determination of* $\boldsymbol{K_2}$ *and* $\boldsymbol{K_3}$

We can proceed by a state transformation :

$$z_{1k} = X_k \, , z_{2k} = X_k^{[2]}, z_{3k} = X_k^{[3]}$$

$$\begin{cases} Z_k = \begin{bmatrix} X_k \\ X_k^{[2]} \\ X_k^{[3]} \end{bmatrix} = \begin{bmatrix} z_{1k} \\ z_{2k} \\ z_{3k} \end{bmatrix} \\ Z_{k+1} = \begin{bmatrix} X_{k+1} \\ X_{k+1}^{[2]} \\ X_{k+1}^{[3]} \end{bmatrix} = \begin{bmatrix} z_{1,k+1} \\ z_{2,k+1} \\ z_{3,k+1} \end{bmatrix} \end{cases} \tag{9}$$

The simplification of the redundant components by the way cited in Appendix leads to these equations :

$$X_{k+1} = (A_1 - BK_1)X_k + (A_2 - BK_2)T_2^+ \tilde{X}_k^{[2]} + (A_3 - BK_3)T_3^+ \tilde{X}_k^{[3]} \tag{10}$$

when considering the new state variables, one can write:

$$z_{1,k+1} = (A_1 - BK_1)z_{1k} + (A_2 - BK_2)T_2^+ \tilde{z}_{2k} + (A_3 - BK_3)T_3^+ \tilde{z}_{3k}$$

$$X_{k+1}^{[2]} = \left[(A_1 - BK_1)X_k + (A_2 - BK_2)T_2^+ \tilde{X}_k^{[2]} + (A_1 - BK_1)^{[3]} T_3^+ \tilde{X}_k^{[3]}\right]^2$$

yields

$$z_{2,k+1} = [A_1 - BK_1]^2 T_2^+ \tilde{z}_{2k} + [(A_1 - BK_1) \otimes (A_2 - BK_2) + (A_2 - BK_2)(A_1 - BK_1)]T_3^+ \tilde{z}_{3k}$$

$$X_{k+1}^{[3]} = \left[(A_1 - BK_1)X_k + (A_2 - BK_2)T_2^+ \tilde{X}_k^{[2]} + (A_1 - BK_1)^{[3]} T_3^+ \tilde{X}_k^{[3]}\right]^3 = (A_1 - BK_1)^{[3]} T_3^+ \tilde{X}_k^{[3]}$$

$$z_{3,k+1} = (A_1 - BK_1)^{[3]} T_3^+ \tilde{z}_{3k}$$

A procedure of the determination of the matrix T_j has been proposed in [19].

The vector Z_{k+1} can be written as following :

$$\begin{bmatrix} z_{1,k+1} \\ z_{2,k+1} \\ z_{3,k+1} \end{bmatrix} = \mathcal{A} \begin{bmatrix} z_{1k} \\ z_{2k} \\ z_{3k} \end{bmatrix} \tag{11}$$

with

$$\mathcal{A} = \begin{bmatrix} A_1 - BK_1 & (A_2 - BK_2)T_2^+ & (A_3 - BK_3)T_3^+ \\ 0 & (A_1 - BK_1)^{[2]} T_2^+ & [(A_1 - BK_1) \oplus (A_2 - BK_2)]T_3^+ \\ 0 & 0 & (A_1 - BK_1)^{[3]} \end{bmatrix}$$

The optimization of the initial studied system (1) is then reduced to the optimization of the new following controlled system

$$Z_{k+1} = \mathcal{A} \; Z_k \tag{12}$$

associated to the control law :

$$U_k = [K_1 \quad K_2 \quad K_3] \, Z_k \tag{13}$$

Based on (9) and substituting U_k by (13) in the quadratic cost J (2), one obtains:

$$J = \sum_{k=0}^{\infty} X_k^T Q X_k - ([K_1 \quad K_2 \quad K_3] Z_k)^T R [K_1 \quad K_2 \quad K_3] Z_k$$

$$J = \sum_{k=0}^{\infty}([I_n \quad 0 \quad 0]Z_k)^T Q [I_n \quad 0 \quad 0]Z_k$$
$$-Z_k{}^T[K_1 \quad K_2 \quad K_3]^T R[K_1 \quad K_2 \quad K_3]Z_k$$
$$J = \sum_{k=0}^{\infty} Z_k^T \boldsymbol{Q}\ Z_k \qquad (14)$$

where

$$\boldsymbol{Q} = [I_n \quad 0 \quad 0]Q\,[I_n \quad 0 \quad 0] - [K_1 \quad K_2 \quad K_3]^T R[K_1 \quad K_2 \quad K_3]$$

III. ILLUSTRATIVE EXAMPLE

Consider the following nonlinear discrete time system:

$$\begin{cases} x_1(k+1) = x_1(k) - x_2(k) + 0.5x_1^2(k) \\ \qquad +0.5x_3^2(k) + u_1(k) \\ x_2(k+1) = x_1(k) + x_2(k) - 0.5x_2^2(k) \\ \qquad +0.5x_1^3(k) + u_2(k) \end{cases} \qquad (15)$$

By linearizing this system around the origin we obtain a linear system described by the linear following representation :

$$\begin{cases} x_1(k+1) = x_1(k) - x_2(k) + u_1(k) \\ x_2(k+1) = x_1(k) + x_2(k) + u_2(k) \end{cases} \qquad (16)$$

where k is the discrete time step, and $u(k)$ is the control law of system which is expressed by (6) in which K_1 is the solution of the Riccati equation (8).

$$K_1 = [0.9023 \quad 0.4437]$$

The initial state is $X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the weighting matrices Q and R are taken :

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \mathrm{R} = 1\,.$$

The quadratic cost of the linearized system is equal :

$$J_2 = 2.9951$$

K_2 and K_3 introduced by (3) are computed by programming the matrix $\mathcal{A}$ expressed in (12) :

$$K_2 = [\ 0.1096 \quad -0.0077 \quad -0.0077 \quad -0.7130]$$

$$K_3 = [0.3376 \quad 0.0572 \quad 0.0572 \quad 0.0239 \quad 0.0572 \quad 0.0239 \quad 0.0239 \quad -0.1401]$$

The quadratic cost (14) is calculated :

$$J_1 = 2.8647$$

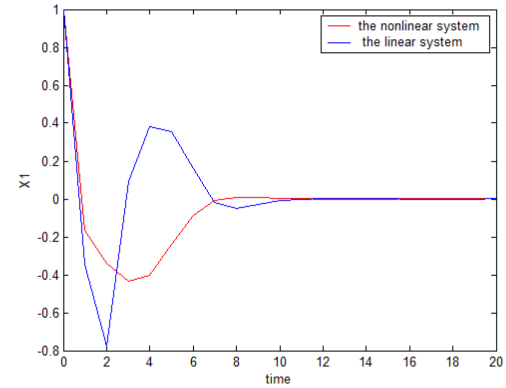

Fig1. Comparison of the first state x_1 of the nonlinear (15) and linearized systems (16)

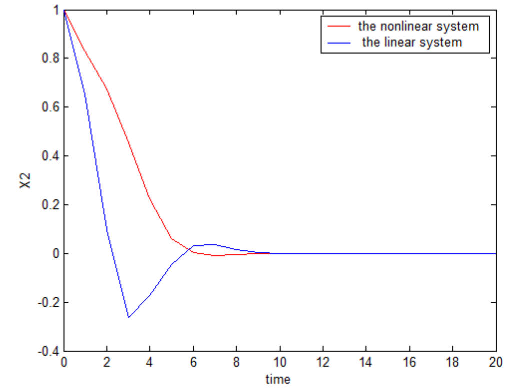

Fig 2. Comparison of the second state x_2 of the nonlinear (23) and linear systems (24)

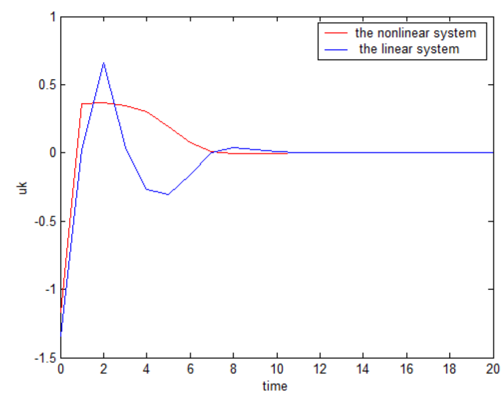

Fig 3. Comparison of the control law of the nonlinear and linear system

The simulation results figure 1-3 show a comparative evolution of the state variables (x_1 ,x_2) and the control law of the nonlinear studied system and the linearized one

the nonlinear control makes the states converge smoothly and rapidly to the origin.

The simulation results show that the criterion value is less for the nonlinear polynomial system described in (15). Thus, the second order polynomial system controls the system variables significantly better than the linear one from both points of view. The obtained results prove that th best gain matrix based on the linearized model could still far from achieving the optimal performance.

IV. CONCLUSION

In this paper presents a technique for solving a class of nonlinear optimal control problem is presented. It is based on the use of notations and properties of the tensor product , the nun-redundant state formulation and the polynomial description of the studied system The different steps of this procedure have been described in order to determine the new structure of optimal control. All the matrices are computed by a numerical procedure implemented in Matlab. The simulation study shows the efficiency of this approach.

APPENDIX

The dimensions of the matrices used here are the following :

$A(p \times q)$, $B(r \times s)$, $C(q \times g)$, $D(s \times h)$,

$E(n \times p)$, $X(n \times 1) \in \Re^n$.

* The kth row of a matrix such as A is denoted $A_{k.}$ and the kth column is denoted $A_{.k}$. The ik element of A will be denoted a_{ik}

* The Kronocker product of A and B is denoted $A \otimes B$ a $(p.r \times q.s)$ matrix, and the $i-th$ Kronoker's power of A denoted $A^{[i]} = A \otimes A \otimes ... \otimes A$ is a $(p^i \times q^i)$ matrix

* The nun-redundant j-power $\tilde{X}^{[j]}$ of the state vector X defined as :

$$X^{[1]} \in \Re^n$$

$$\tilde{X}^{[1]} = X^{[1]} = X$$

$$\begin{cases} \forall j \geq 2 \; \tilde{X}^{[j]} = [x_1^j, x_1^{j-1}x_2, \dots, x_1^{j-1}x_n, x_1^{j-2}x_2^2 \\ \qquad , x_1^{j-2}x_2x_3, \dots, x_1^{j-1}x_2x_n, \dots, \\ \qquad x_1^{j-2}x_n^2, \dots, x_1^{j-3}x_2^3, \dots, x_n^j]^T \end{cases}$$

where the repeated components of the redundant j-power $X^{[1]}$ are omitted. Then we have the following relation

$$\begin{cases} \forall j \in \mathcal{N} \;\; \exists! \; T_j \in \Re^{n^j \times \alpha_j}; \; \alpha_j = \binom{n+j-1}{j} \\ X^{[j]} = T_j \tilde{X}^{[j]} \end{cases}$$

Thus, one possible solution for the inversion can be written as the following form :

$$\tilde{X}^{[j]} = T_j^+ X^{[j]}$$

Where T_j^+ is the Moore-Penrose pseudo inverse of T_j given by : $T_j^+ = (T_j^T T_j) T_j^T$ and α_j stands for the binomial coefficients.

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