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A Novel Swing-up Control of the Inverted Pendulum by Brunovsky Trajectory Tracking Method

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***Abstract*— The Inverted Pendulum (IP) is a standard benchmark often used to test non-linear control strategies. The swing-up control of the IP should bring-up the pendulum from its downward stable position to its unstable upright position. The majority of researchers have focused on non-linear swing-up of the IP based on energy methods. This paper presents a novel non-linear swing-up control of the IP by the Brunovsky trajectory tracking method. In this method the trajectory of the control variable which is the pendulum angle is chosen first, and using the dynamical non-linear model of the IP the control law is then derived. Simulation results show a suitable swing-up of the IP by the proposed method.**

***Keywords*— Inverted Pendulum, Non-linear, Swing-up, Brunovsky, Trajectory tracking.**

I. Introduction

Both cart inverted pendulum and the rotary inverted pendulum called Furuta pendulum have been widely used in the literature as benchmarks for testing non-linear control strategies.
The inverted pendulum (IP) considered in the paper is mounted on a cart, the rotational angle of the pendulum and the horizontal cart position are both controlled by a force applied to the cart.
The pendulum should be brought up from its downward stable position to its unstable upright position; this is the swing-up control. A regulation strategy must then be employed to stabilize the pendulum on its vertically upward position.

Since the IP is an underactuated, multivariable, highly non-linear and unstable system, it was commonly used as a benchmark to test linear and non linear control strategies. While the swinging-up phase of the IP is non-linear its stabilization control could be linearised around the vertically upward position.
Different control strategies have been proposed in the literature for both swinging-up and stabilisation control of the IP. The most interesting ones for the swing-up phase are the energy based control strategies [1], [2], [3], [4], methods using the restriction of the cart track length [5], [6], [7], and intelligent control techniques [8], [9], [10]. For the stabilisation control, in general, classical linear control methods are applied after linearization of the IP model around its unstable equilibrium [10], [11], [12].

In this paper, a new swinging-up strategy is proposed based on the Brunovsky tracking trajectory method. It consists in defining the initial and final states of the pendulum, using a specified function for the control angle, and has the advantage to balance the pendulum as fast as possible. When the pendulum reaches the upper vertical a linear PID control is then used to stabilise it at its unstable equilibrium.

II. The Model of the Cart Inverted Pendulum

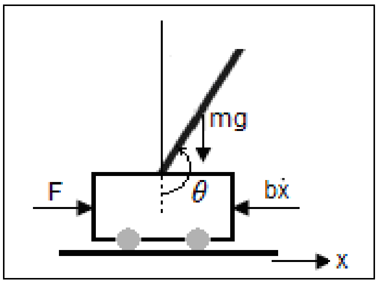

Fig. 1 Cart Inverted Pendulum system

The inverted pendulum shown in fig1 is mounted on a moving cart actuated via a control force F, the displacement of the cart induces the rotation of the pendulum.
The system motion is described by the following differential equations [4]:

$$(M+m)\ddot{x}+b.\dot{x}+m.l.\ddot{\theta}\cos(\theta)-m.l.\dot{\theta}^2.\sin(\theta)=F \quad (1)$$

$$(I+m.l^2).\ddot{\theta}+m.g.l.\sin(\theta)=-m.l\ddot{x}.\cos(\theta) \quad (2)$$

Where:
- x, $\dot{x}$ and $\ddot{x}$ are respectively the cart displacement, velocity and acceleration.
- θ, $\dot{\theta}$ and $\ddot{\theta}$ are respectively the pendulum angle, velocity and acceleration.
- M and m are respectively the cart and pendulum weights.
- l is the pendulum length.

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III. Proposed Swing-up Strategy by Brunovsky Trajectory Tracking Method

The swing-up problem of the IP is to determine the control law which could bring-up the pendulum from its initial equilibrium where $\theta = 0$ to its upright vertical position which corresponds to $\theta = \pi$.

The swing-up control law u(t) is derived from (1).

$$u(t) = (M + m).\ddot{x} + b.\dot{x} + m.l.\ddot{\theta}\cos(\theta) - m.l.\dot{\theta}^2.\sin(\theta) \qquad (3)$$

In the Brunovsky trajectory tracking method [13] the control variable θ can be chosen as given by (6), the initial and final conditions of θ and its derivatives are well known, and ϕ is a C^4 function given by equation (4).

Replacing the pendulum angle θ, velocity $\dot{\theta}$, and acceleration $\ddot{\theta}$ by their initial and final values as mentioned in table1, and $\phi(t)$ by its expression (4) where σ is $\frac{t}{T}$, the expression of θ is hence done by (5).

$$\phi(\sigma) = \frac{\sigma^4}{\sigma^4 + (1-\sigma)^4} \qquad (4)$$

TABLE I
IMPOSED INITIAL AND FINAL CONDITIONS
of θ AND ITS DERIVATIVES

	$\theta(t)$	$\dot{\theta}(t)$	$\ddot{\theta}(t)$	$\dddot{\theta}(t)$
IC	0	0	0	0
FC	Π	0	0	0

$$\theta(t) = \frac{\pi.t^4}{T^4.\left(\frac{t^4}{T^4} + \left(1 - \frac{t}{T}\right)^4\right)} \qquad (5)$$

The expressions of $\dot{\theta}$ and $\ddot{\theta}$ are then derived as follows:

$$\dot{\theta} = \frac{4.\pi.t^3.T.(T-t)^3}{\left(2.t^4 + T^4 - 4.T^3t + 6T^2t^2 - 4T.t^3\right)^2} \qquad (7)$$

$$\ddot{\theta} = \frac{4.\pi.t^2T.(T-t)^2.\left(-10.T.t^4 + 3.T^5 - 10.T^4t + 10.T^3t^2 + 4t^5\right)}{\left(2t^4 + T^4 - 4.T^3t + 6T^2t^2 - 4T.t^3\right)^3} \qquad (8)$$

$$\ddot{x} = -\frac{(I + m.l^2).\ddot{\theta} + m.g.l.\sin(\theta)}{m.l.\cos(\theta)} \qquad (9)$$

Substituting the angle θ and the acceleration $\ddot{\theta}$ by their expressions in equation (9) we obtain the expressions of the cart acceleration $\ddot{x}$ and velocity $\dot{x}$ (10).

$$\theta(t) = \left(1 - \phi(\frac{t}{T})\right).\left[\theta_o + t.\dot{\theta}_o + \frac{t^2}{2}.\ddot{\theta}_o + \frac{t^3}{6}.\dddot{\theta}_o\right] + \phi(\frac{t}{T}).\left[\theta_T + (t-T).\dot{\theta}_T + \frac{(t-T)^2}{2}.\ddot{\theta}_T + \frac{(t-T)^3}{6}.\dddot{\theta}_T\right] \qquad (6)$$

$$\dot{x} = \int\left(-\frac{\frac{4.(I + m.l^2)\pi.t^2T.(T-t)^2.\left(-10.T.t^4 + 3.T^5 - 10.T^4t + 10.T^3t^2 + 4t^5\right)}{\left(2.t^4 + T^4 - 4.T^3t + 6T^2t^2 - 4T.t^3\right)^3} + m.g.l.\sin\left(\frac{\pi.t^4}{T^4.\left(\frac{t^4}{T^4} + \left(1 - \frac{t}{T}\right)^4\right)}\right)}{m.l.\cos\left(\frac{\pi.t^4}{T^4.\left(\frac{t^4}{T^4} + \left(1 - \frac{t}{T}\right)^4\right)}\right)}\right).dt \qquad (10)$$

Determination of $\dot{x}$:

The expression of $\dot{x}$ is to be found by integrating $\ddot{x}$, since the expression of $\ddot{x}$ given in (10) is complicated and has a singularity at $\theta = \frac{\pi}{2}$ which corresponds to $t = \frac{T}{2}$, Maple software is used in integration. Unfortunately the numerical tool could not give an analytical expression for $\dot{x}$, the solution to this problem is to sample $\ddot{x}$. Another problem is encountered while integrating is that $\ddot{x}$ is infinite at $\theta = \frac{\pi}{2}$, so the software could not integrate it at $t = \frac{T}{2}$ and for times later. Since $\ddot{x}$ is symmetric as shown in fig3, its integration in the interval $\left[\frac{T}{2}^{+} \; T\right]$ is deduced from its calculus in the interval $\left[0 \; \frac{T}{2}^{-}\right]$.

It was proven that to have an accurate swing-up of the inverted pendulum, the simulation should be run with a variable step, so the control law should be continuous.
When making a brief review of how the control signal u(t) (3) was determined, we notice that all functions seen in equation (3) are continuous excepted $\dot{x}$, we used as a solution to this problem the Matlab function 'spline' to interpolate the samples of $\dot{x}$.

IV. Simulation Results

The proposed swing-up method explained above is simulated in Matlab Simulink.
The simulation parameters of the pendulum model are: M= 0.5 kg, m=0.2kg, l=0.3m, g=9.8 $m.s^{-2}$, I=0.06 $kg.m^2$ and k=0.1 $kg.m^2s^{-1}$.
The control law u(t) calculated in the previous section is shown in fig4. System outputs which are the pendulum angle θ, velocity $\dot{\theta}$, the cart displacement x and velocity $\dot{x}$ are shown in fig5.

To show the effectiveness of the swing-up method a PID controller was added to the proposed controller in order to stabilise the IP at its unstable position.
As seen in fig5 the pendulum swings-up from its stable download position where the angle $\theta = 0$ to its unstable upright position where $\theta = \pi$. The curve of the angular position of the IP has an inflection point at approximately half swinging time, which corresponds to t=1s and $\theta = \frac{\pi}{2}$, at this time the angular velocity $\dot{\theta}$ has a maximum.

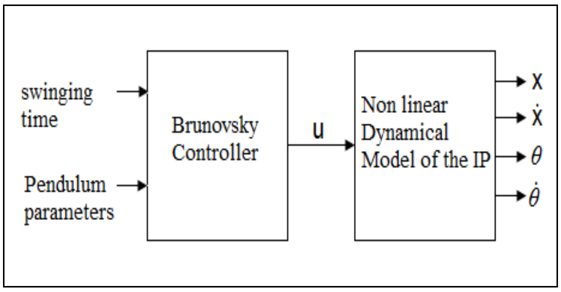

Fig. 2 Block Diagram of the open loop Brunovsky Controller

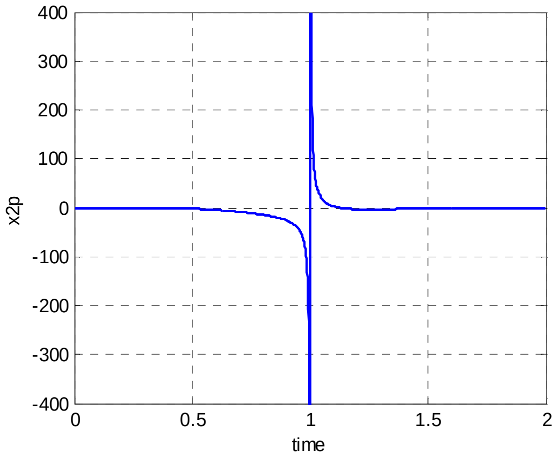

Fig. 3 Plot of $\ddot{x}$ zoomed (at t=1s x2p level is infinite)

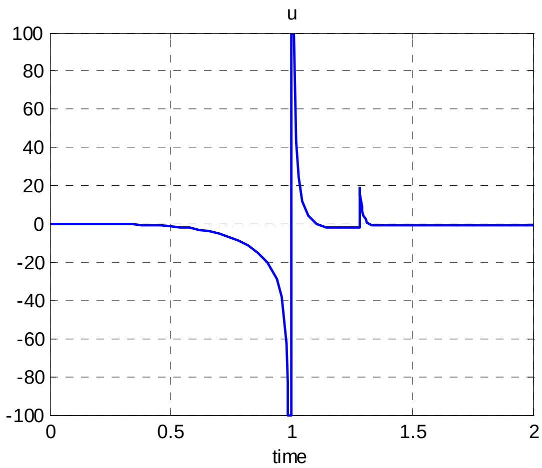

Fig. 4 Control law u with saturation

The pendulum reaches the upper vertical with a zero velocity which allows to stabilise the pendulum at $\theta = \pi$.
For a better illustration of the pendulum angle trajectory the curves of θ cosines and sinuses are plotted in fig7, their waveforms confirm that the pendulum swings-up from $\theta = 0$ to $\theta = \pi$.
Concerning the cart displacement x and velocity $\dot{x}$, we notice from fig6 that the cart is freely moving, it has only to give the sufficient energy to swing-up the pendulum. An additive control could be made to control and stabilise the cart which is not in the scope of this paper.

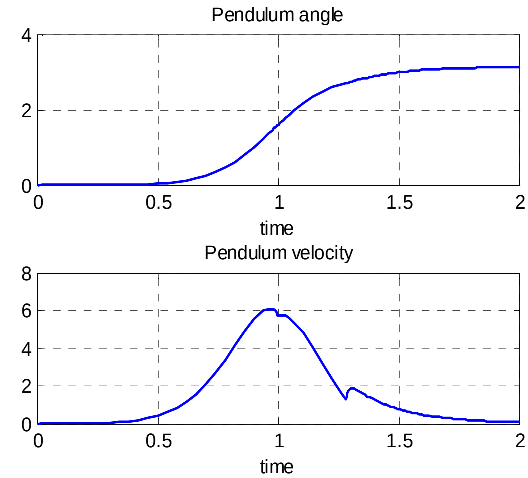

Fig. 5 Pendulum angle(θ) and velocity ($\dot{\theta}$)

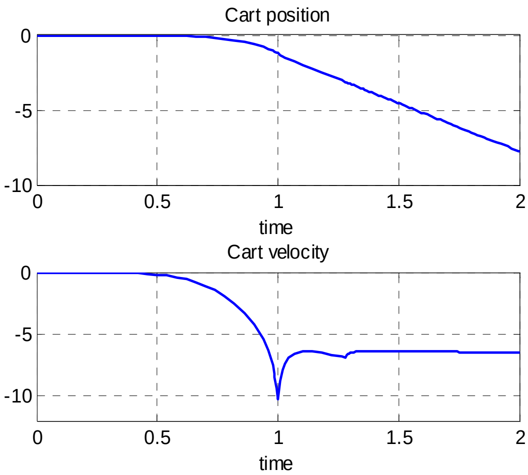

Fig. 6 Cart displacement(x) and velocity ($\dot{x}$)

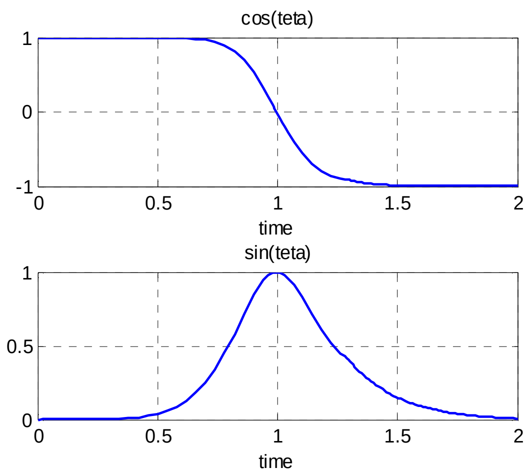

Fig. 7 Cosinus and sinus of θ

V. CONCLUSION

In this paper a novel swing-up control of the inverted pendulum IP is proposed, it is a non-linear open-loop control based on the Brunovsky trajectory tracking method. This method allows determining the control law that swings-up the pendulum from its download stable position to its upright unstable position for a given swinging time. After defining the initial and final conditions of the pendulum angle, and using some Brunovsky function the control law is calculated. The swinging of the pendulum is done by a free displacement of the cart; a stabilisation control should then be done for both pendulum and cart. The control strategy proposed in this paper could be applied to other non-linear dynamical systems.

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