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Computation of the Zeros of the Mittag-Leffer as Bargmann Functions

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Abstract

The Computation of the zeros of the function are of fundamental importance and play a significant roal in the dynamic solution and it has mention its application in many scientific areas from mathmatics, physics and communication The aim of this paper is to computation the number of zeros of the solution of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ as examples of Bargmann function with arbitrary order of growth. We find that the Mittag-Leffler function have not only the same type of zeros. The number of zeros cab be any finite number:1,2,3,..., not necessarily an odd number.

Keyword

Bargmann analytic representation , Mittag-Leffler Function , Zeros of Function.

1 Introduction

The general theory of growth of an analytic function and the density of their zeros, is applied to the Bargmann function.As example of the Bargmann function it has used the Mittag-Leffler function $E_{\alpha,\beta}(z)$ for the arbitrary complex argument Z, and two parameters $\alpha, \beta \in \mathbb{R}$.The Mittag-Leffler function plays an important role in Mathematica model, its first formulation by the Swedish Mathematician [3] Magnus Gsta Mittag-Leffler $(1846 - 1927)$, The function became a relevant topic, not only from the pure mathematical point of view, but also from the perspective of its applications.The special function in this case is:

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}, \qquad \alpha{>}0, z \in \mathbb{C}$$

And its general form.

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}. \qquad \alpha{>}0, \beta, z \in \mathbb{C}$$

In this paper we give a whole clarified description for the zeros of the solution of the Mittag-Leffler function $E_{\alpha,\beta}(z)$,$0 < \alpha < 2$. We find that the number of zeros can be any finite number :1,2,..., not necessarily an odd number. The paper is organized as follows.Section 2 studied the Bargmann analytic representation and their growth of these function. Section 3 introduces the fundamental aspects of The Mittag-Leffler function and their States as Bragmann function. Section 4 The zeros of the Mittag-Leffler function as Bragmann function, are considered. Finally, Section 5 outlines the main conclusions

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2 The Bargmann analytic representation and Their growth.

There are several representation that use analytic function.The Bargmann representation is the most well-known one.In this section,we introduce the Bargmann analytic representation in the complex plane defined by the Glauber coherent state. The space of these functions is defined as the space of the entire functions with no singularities. The growth of an analytic function is characterised by its order and type [1], [4], [5],[14], [15], [16]. Let $|k\rangle$ be an arbitrary state :

$$|k\rangle = \sum_{n=0}^{\infty} k_n |n\rangle. \tag{1}$$

The normalization condition is given below

$$\sum_{n=0}^{\infty} |k_n|^2 = 1. \tag{2}$$

The conjugate of $|k\rangle$ is $\langle k|$ can be written as follows:

$$\langle k| = \sum_{n=0}^{\infty} k_n^* \langle n| \ ; \ |k\rangle^* = \sum_{n=0}^{\infty} k_n^* |n\rangle. \tag{3}$$

The Bargmann representation ,[2],[6], for the state $|k\rangle$ is represented by :

$$k(z) = exp(\frac{|z|^2}{2})\langle z^*|k\rangle = \sum_{n=0}^{\infty} \frac{k_n z^n}{\sqrt{n!}}. \tag{4}$$

- The Bargmann representation for the number states $|n\rangle$ is:

$$k(z) = \frac{z^n}{\sqrt{n!}} \tag{5}$$

- The Bargmann representation for the coherent state $|A\rangle$ is:

$$k(z) = exp(Az - \frac{|A|^2}{2}) \tag{6}$$

Which is of order $\rho = 1$
and type $\tau = |A|$.

•The Bargmann function of the squeezed state$|A; r, \theta, \lambda\rangle$ is :

$$k(z) = (1 - |\tau|^2)^{1/4} \exp[\frac{\tau}{2} z^2 + \beta z + \lambda] \tag{7}$$

$$\tau = -\tanh(\frac{1}{2}r)\exp(-i\theta) \ , \ \beta = A(1 - |\tau|^2)^{1/2}$$
$$, \ \lambda = -\frac{1}{2}\tau^* A^2 - \frac{1}{2}|A|^2. \tag{8}$$

It has growth with order $\rho = 2$ and type $\tau = \frac{1}{2}\tanh(\frac{1}{2}r)$.

- State with the Mittag-Leffler

function as Bargmann function $|\rho, \tau\rangle$ is:

$$|\rho, \sigma\rangle = \sum_{N=0}^{\infty} \frac{\sigma^{\frac{N}{\rho}} (N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho} + \beta)} * [\sum_{N=0}^{\infty} \frac{\sigma^{\frac{2N}{\rho}} (N!)}{\Gamma(\frac{N}{\rho} + \beta)}]^{\frac{-1}{2}} |N\rangle \tag{9}$$

when $0 \leq \rho < 2$; and also when $\rho = 2$ and $\tau < \frac{1}{2}$, It has growth with order $\rho = \frac{1}{\alpha}$ and type τ for any β .

3 The Mittag-Leffler function as Bragmann function.

In this section it has introduce the Mittag-Leffler function as Bargmann function as example where the order of growth is fractional. The zeros of the Mittag-Leffler function are studied.

3.1 The Mittag-Leffler function

The Mittag-Leffler function is named after the great Swedish mathematician Gosta Magnus Mittag-Leffler (1846-1927). He has worked on the general theory of functions, studying the relationship between independent and dependent variables.The generalization of the Mittag-Lefer function was proposed by Wiman in his work [10] on zeros of function which is defined by the series :

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}, \qquad \alpha > 0, z \in \mathbb{C} \tag{10}$$

More generally, the Mittag-Leffler function with two parameters has the form:

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}. \qquad \alpha > 0, \beta, z \in \mathbb{C} \tag{11}$$

Here z is a complex variable and α, β are arbitrary positive constants. The function $E_{\alpha,\beta}(z)$ is an entire function of the complex variable z.
Kilbas et al. studied the generalized Mittag-Leffler function with three parameters [7]. This function was also introduced

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by T.R Prabhakar in 1971[13]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m(\gamma)_m}{m!\Gamma(\alpha m+\beta)}, \qquad (\gamma)_m = \frac{\Gamma(m+\gamma)}{\Gamma(\gamma)}, \quad (12)$$

where α, β andγ are arbitrary positive constants, and $(\gamma)_L$ is the Pochhammer symbol[13].

In some other applications, a generalized Mittag-Leffler function has four parameters, the following function was introduced by Dzherbashian [13], and is defined as follows :

$$E^{\gamma,\eta}_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m(\gamma)_{m\eta}}{\Gamma(\alpha m+\beta)m!}, \quad (13)$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\eta \in \mathbb{N}$. When $(\gamma)_0 = 1$ and $(\gamma)_m = \frac{\Gamma(m+\gamma)}{\Gamma(\gamma)}$.

In the next section we provide details of some special properties of the Mittag-Leffler function.

3.2 Analytic properties of the Mittag-Lefer function

First of all, we have to mention that for $Re\alpha > 0$ and arbitrary complex parameter β, the Mittag-Lefer function $E_{\alpha,\beta}(z)$ is an entire function of the complex variable z. For particular values of parameters, the Mittag-Lefer function coincides with some elementary functions. A description of the most important properties of this function can be found in the third volum of the Bateman project [7],[8],[11],[9],[13]. In this case when using their series representations for some parameters. It is easy to see that:

$$E_{0,1}(z) = \frac{1}{1-z}, |z| < 1; \quad (14)$$

$$E^{1,1}_{\alpha,1}(z) = E^{1}_{\alpha,1}(z) = E_{1,1}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+1)} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = e^z \quad (15)$$

$$E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = E_{1,2}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+2)} = \frac{e^z-1}{z} \quad (16)$$

$$E_{2,1}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(2m+1)} = \sum_{k=0}^{\infty} \frac{z^m}{2m!} = \cosh(\sqrt{z}) \quad (17)$$

$$E_{2,2}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(2m+2)} = \frac{sinh(\sqrt{z})}{\sqrt{z}} \quad (18)$$

And in general we can show some of this function as follows:-

$$\begin{aligned} E_{1,n}(z) &= \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+n)} = \sum_{m=0}^{\infty} \frac{z^m}{(m+(n-1))!} \\ &= \frac{1}{z^{n-1}} \sum_{m=0}^{\infty} \frac{z^{m+(n-1)}}{(m+(n-1))!} \\ &= \sum_{m=0}^{\infty} \frac{1}{z^{n-1}}[e^z - \sum_{m=0}^{n-2} \frac{z^m}{m!}], \quad n = 1,2,... \end{aligned} \quad (19)$$

If $\alpha, \beta > 0$ then they takes the formula[11].

$$z^t E_{\alpha,\beta+t\alpha}(z) = E_{\alpha,\beta}(z) - \sum_{m=0}^{t-1} \frac{z^m}{\Gamma(m\alpha+\beta)}, \qquad t \in \mathbb{N}. \quad (20)$$

As example if $t = 1$ we get:

$$z^1 E_{\alpha,\beta+1\alpha}(z) = E_{\alpha,\beta}(z) - \frac{z^0}{\Gamma(0\alpha+\beta)} = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)}. \quad (21)$$

3.3 The Mittag-Leffler states: States with the Mittag-Leffler function as Bargmann function

In this section we extend the construction of the states which has the Bargmann function with a given order ρ (ρ can be any values between 0 and 2 , also it can be $\rho > 2$ but then the function is not normalizable) and given type σ by choosing the coefficients $\mathcal{C}_N$ in:

$$|\rho,\sigma\rangle = \sum_{N=0}^{\infty} \mathcal{K}_N|N\rangle; \qquad \mathcal{K}_N = L\mathcal{C}_N; \qquad \mathcal{C}_N = \frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+\beta)}. \quad (22)$$

$$\mathcal{K}_N = L\frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+1)}, \quad (23)$$

where L is a normalization constant given by the following:-

$$L = [\sum_{N=0}^{\infty} \frac{\sigma^{\frac{2N}{\rho}}(N!)}{[\Gamma(\frac{N}{\rho}+1)]^2}]^{\frac{-1}{2}}, \quad (24)$$

then we can write:

$$|\rho,\sigma\rangle = \sum_{N=0}^{\infty} \frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+\beta)} * [\sum_{N=0}^{\infty} \frac{\sigma^{\frac{2N}{\rho}}(N!)}{\Gamma(\frac{N}{\rho}+\beta)}]^{\frac{-1}{2}}|N\rangle \quad (25)$$

L is finite when $0 \leq \rho<2$; and also when $\rho = 2$ and $\sigma < \frac{1}{2}$. The Bargmann function of this state $LE_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}}z)$ where $E_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}}z)$ is the Mittage-Leffer function.

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Vol.38 pp.9-14

and we can writ it as follows:-

$$K(z) = \sum_{N=0}^{\infty} \mathcal{K}_N z^N (N!)^{\frac{-1}{2}}, \tag{26}$$

inserting $Eq(23)$ in equation $Eq(26)$ we get :

$$K(z) = \sum_{N=0}^{\infty} L \frac{\sigma^{\frac{N}{\rho}} (N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho} + \beta)} z^N (N!)^{\frac{-1}{2}} = \sum_{N=0}^{\infty} L \frac{(\sigma^{\frac{1}{\rho}} z)^N}{\Gamma(\frac{N}{\rho} + \beta)} = L E_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}} z). \tag{27}$$

We confine $z \in \mathbb{C}$ and consider the zeroes of the function $K(z)$ in Eq 27, where $E_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}} z)$ is the Mittag-Leffler function when $\beta = 1$. As an example we present extensive numerical calculation of the function $K(z)$ in the complex plan. We show a three-dimensional plot of the real and imaginary parts. ρ can take all values between 0 and 2. We have considered numerical results are presented in Fig $(1), (2)$. The total numbers of the zeros in this case can be easily enumerated. In the special case where $\rho = 1$ the state Eq (25) is reduced to usual coherent states and when $\rho = 2$ the state Eq (25) is reduced to squeezed states.

4 The Zeros of the Mittag-Leffler function as Bargmann function

An entire function of fractional order can have infinitely many zeros. Also there are entire function which have few zero or no zeroes , (eg..the exponential function [14], [16], [17],[18], [19], [20],[21]). The Mittag-Leffler function which was given in Equations (10) and,(11) is an entire function of order $\frac{1}{\alpha}$ Consequently, Mittag-Leffler function $E_{\alpha,\beta}(z)$ might have an infinite number of zeros with the possible exception when $\frac{1}{\alpha}$ is an integer. In this case, there may be a finite number of zeros, or an infinite number of zeros. We can show that with the exception of $\alpha = \beta = 1$,the Mittag-Leffler function has an infinite number of zeros[20].

The Mittag-Leffler function $E_{(1,1)}(z)$ is equal to the exponential function e^z and is only the function which has no zeros. In this section we calculate the zeros of polynomial approximations to $E_{\alpha,\beta}(z)$ using Eq (11) when $\beta = 1$ can be $2; 4; 6; ....$ we demonstrate this procedure numerically for α increasing from $1.4 < \alpha < 1.99$ and where z is real. In Fig (3),(4),and (5) we plot some curves of $E_{\alpha,1}(z)$. For example when the $\alpha = 1.567$, $E_{\alpha,1}(z)$ curve crosses the x-axis four times yielding two zeros, the next larger value of α when $\alpha = 1.759$, $E_{\alpha,1}(z)$ has six zeros, and when $\alpha = 1.957$,$E_{\alpha,1}(z)$ has more than ten zeros.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$, which is a generalization of the exponential and trigonometric functions, arises frequently in problems of fractional calculus and hence, to understand the theory of fractional differential equations, one needs to understand properties of this function. One property which is of interest is the nature of its zeros. The main results regarding zeros of $E_{\alpha,\beta}(z)$ when α is a real number lying between 1 and 3 may be summarized as follows: When $1 < \alpha < 2$, there is a finite number (possibly zero) of real zeros and an infinity of complex zeros. When $2 < \alpha \leq 3$, there are a finite number (possibly zero) of complex zeros and an infinite number of real zeros. The number of complex zeros goes as $\log \beta$ and the complex zeros are contained in a small region near the origin[17].

Remark : The exponential function $E_{1,1}(z) = \exp(z)$ is the only Mittag-Leffler function which has no zeros. All other function $E_{\alpha,\beta}(z)$ $Re\alpha > 0, \alpha \neq 1$ have infinitely zeros. For example, for $\alpha, \beta = 1$, has no zeros, but its polynomial approximation,has N zeros:

$$e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^N}{N!} = \sum_{N=0}^{\infty} \frac{z^N}{N!}, \tag{28}$$

5 Conclusion

In this paper, we stated The Bargmann analytic representation in the complex plane. We also considered the growth of the Mittag-Leffler function as Bargmann function, and the parametric and analytic properties of the function . We also considered the zeros of the Mittag-Leffler function when $1 < \alpha < 2$. Furthermore, we considered the Bargmann function of the zeros of the function and we calculated the number of zeros for any value of α in the area of $0 < \alpha < 2$. The number of zeros can be any finite number.

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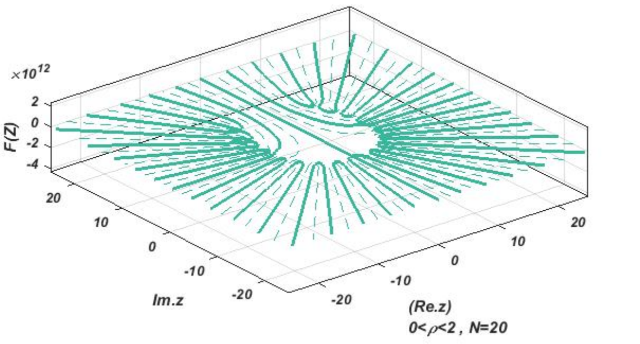

Figure 1: The zeros of the function $K_N(z)$ in Eq (27),with $N = 20$ in (a)for $E_{\alpha,1}(z)$ when $0 < \alpha < 2$, Polynomial approximation have been made, and their zeros may be fictitious.

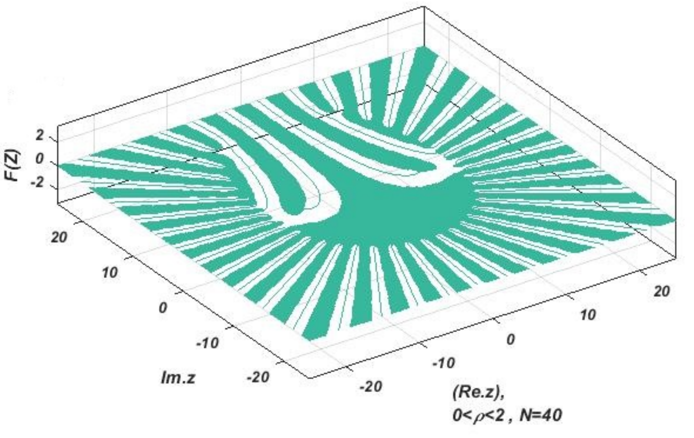

Figure 2: The zeros of the function $K_N(z)$ in Eq (27), with $N =$ for $E_{\alpha,1}(z)$ in (b) when $0 < \alpha < 2$, Polynomial approximation l been made, and their zeros may be fictitious.

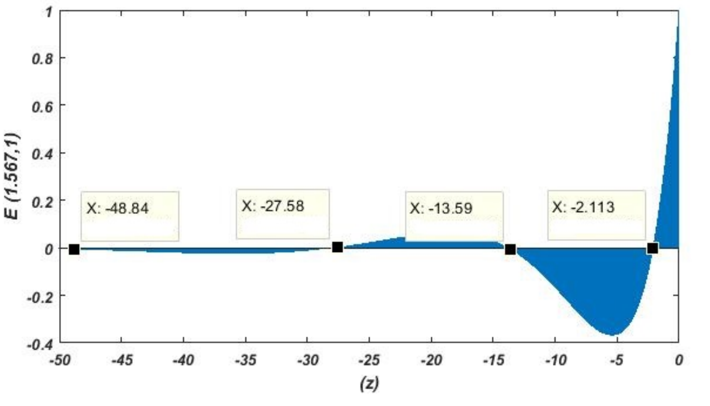

Figure 3: The zeros of the function $E_{\alpha,1}(z)$ when $\alpha = 1.567$.

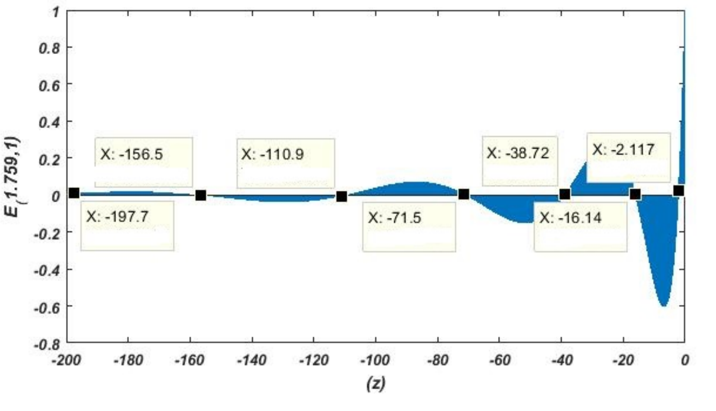

Figure 4: The zeros of the function $E_{\alpha,1}(z)$ when $\alpha = 1.759$.

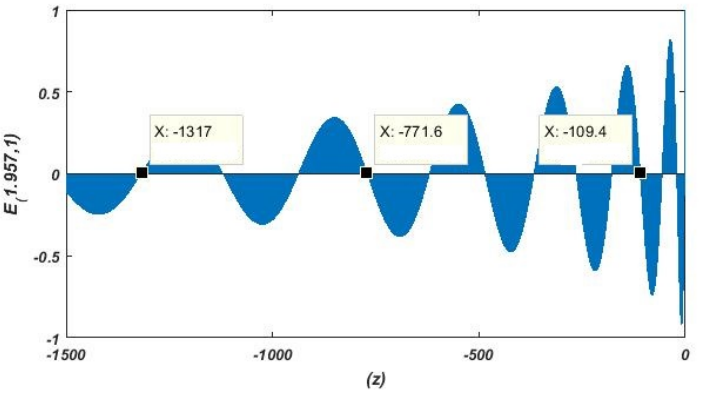

Figure 5: The zeros of the function $E_{\alpha,1}(z)$ when $\alpha = 1.957$.

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