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PSO-based Optimal Control of Linear Switched Systems with Pure Inequality State Constraints

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***Abstract*—This paper is meant to study the optimal control of switched linear systems under inequality state constraints. This study consists in dividing the optimal control problem into two stages. On the one hand, we need to find the optimal input by the Pontryagin principle and more precisely by the Ricatti equation while satisfying the Karush-Kuhn-Tucker (KKT) conditions. On the other hand, a meta-heuristic method, Particle Swarm Optimization (PSO), is needed to find the optimal switching instants.**

Keywords*—*optimal control, switched linear systems, inequality constraints,Ricatti equation, KKT conditions, PSO.

I. Introduction

A switched system is a particular kind of hybrid system that consists of a number of subsystems and a switching law specifying the active subsystems at each time instant [1].

They describe many phenomena in the real-world processes such as power electronics, chemical processes [2], [3], automotive systems and networked control systems [4], [5] and [6]. Their mathematical models very accurately represent the dynamics of various types of systems in different application areas [7] and [8].

Added to that, switched systems may have a linear or nonlinear dynamic that are either autonomous or controlled. Bearing in mind the real aspect of switched systems, their optimal control usually have equality or inequality constraints on the state [9], or the input [10] or even on both of them [11].

In this paper, we study the case of linear controlled switched systems under pure state inequality constraints. To solve the optimal control problem of this type of systems, we have divided the study into two stages : the first one consists in finding the optimal input by applying the Pontryagin principle while respecting the state inequality constraints. This is done through the determination of the Lagrange multipliers and then through the construction of the Lagrangian. In the second stage, we have used the meta-heuristic approach PSO in order to find the optimal switching instants of different subsystems. These instants guarantee the minimization of a determined cost function.

A numerical example is given by the end, to illustrate the effectiveness of the method for linear constrained switched systems.

II. Problem formulation

A. Switched linear systems

Switched linear systems consist in the subsystems $\dot{x} = A_i x + B_i u$ with $i \in I = \{1,2,3,...,M\}$. In order to control a switched system, one needs to choose not only a continuous input but also a switching sequence [12], [13]. A switching sequence in $t \in [t_0, t_f]$ regulates the sequence of active subsystems, and it is defined as follow:

$$\sigma = \left((t_0, i_0),(t_1, i_1),...,(t_k, i_k),...,(t_K, i_K)\right) \qquad (1)$$

where $K \geq 0$; $t_0 \leq t_1 \leq ... \leq t_K$ and $i_k \in I = \{1,2,3,...K+1\}$ for $k = 1,2,3,...,$ and K. Note that (t_k, i_k) indicates that at instant t_k the system switches from subsystem i_{k-1} to i_k which is active during the time interval $[t_k, t_{k+1}]$.

B. Optimal control

Consider a controlled linear switched system with subsystems [14],[15],[16]:

$$\dot{x} = A_i x + B_i u \qquad (2)$$

$$\text{s.t.} \quad g(x) \leq 0 \qquad (3)$$

with $i \in I = \{1,2,3....M\}$. The vector $g(x) = \left[g_1(x),...,g_q(x)\right]^t$ represents the q linear constraints on the state. Assume that a perspecified sequence of active subsystems $(1,2,3,...,k,...,K+1)$ is given. The optimal control problem consists in finding optimal switching instants $t_1,...,t_K$, $(t_0 \leq t_1 \leq ... \leq t_K \leq t_f)$ that permit the minimization of the cost function in general quadratic form:

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$$\left(\begin{array}{c} J=\frac{1}{2}x\left(t_f\right)^t Q_f x\left(t_f\right)+M_f x\left(t_f\right)+W_f \\ +\int_{t_0}^{t_f}\left(\frac{1}{2}x^t Qx+x^t Vu+\frac{1}{2}u^t Ru+Mx+Nu+W\right)dt \end{array}\right) \quad (4)$$

while respecting the inequality state constraints (3).

Initial and final instants t_0 and t_f are given. The corresponding continuous state trajectory x departs from a given initial state $x\left(t_0\right)=x_0$.

Q_f, M_f, W_f, Q, V, R, M, N and W are matrixes with appropriate dimensions.

Q_f, $Q\geq 0$ and $R>0$.

To solve this optimal control, some steps should be followed.

- Construct the Lagrangian :

$$\Lambda=\frac{1}{2}x^t Qx+x^t Vu+\frac{1}{2}u^t Ru+Mx+Nu+W+\lambda^t g \quad (5)$$

where $\lambda=\left[\lambda_1,...,\lambda_q\right]$ is a vector of Lagrange multipliers verifying the Karush-Kuhn-Tucker (*KKT* conditions) [17], [18], [19], [20].

Karush-Kuhn-Tucker conditions:

Assume that x^* is a local optimum, then there exists a vector $\lambda=\left[\lambda_1,...,\lambda_q\right]\in\mathbb{R}^q$ such that :

$$\begin{cases} \frac{\partial\Lambda}{\partial x^*}=0 \\ \frac{\partial\Lambda}{\partial\lambda}=0 \\ g_j(x^*)\leq 0 \quad ;\ \text{j=1,...,q} \\ \lambda_j\geq 0 \quad ;\ \text{j=1,...,q} \\ \lambda_j.g_j(x^*)=0 \quad ;\ \text{j=1,...,q} \end{cases} \quad (6)$$

- Define the augmented Hamiltonian [21],[22]:

$$H=\frac{1}{2}x^t Qx+x^t Vu+\frac{1}{2}u^t Ru+Mx+Nu+W \\ +p^t(A_i x+B_i u)+\lambda^t g \quad (7)$$

$\forall t\in\left[t_i,t_{i+1}\right)$, the state and costate equations can be written as follow:

$$\frac{\partial H_i}{\partial p}=\dot{x} \quad (8)$$

$$-\frac{\partial H_i}{\partial x}=\dot{p} \quad (9)$$

- The optimal input is determined by the stationary condition:

$$\frac{\partial H_i}{\partial u}=0 \quad (10)$$

$$\text{i.e}\quad \begin{array}{l} u_i=-R^{-1}(B_i^t P+V^t)x \\ \quad -R^{-1}(B_i^t S^t+N^t) \end{array} \quad (11)$$

where P and S satisfy the following general Ricatti equations:

$$-\dot{P}=\left(\begin{array}{c} Q+PA_i+A_i^t P \\ -\left(PB_i+V\right)R^{-1}\left(B_i^t P+V^t\right) \end{array}\right) \quad (12)$$

$$-\dot{S}=\left(\begin{array}{c} M+SA_i \\ -\left(SB_i+N\right)R^{-1}\left(B_i^t P+V^t\right) \end{array}\right) \quad (13)$$

with the limit and continuity conditions:

$$P\left(t_f\right)=Q_f \quad (14)$$

$$S\left(t_f\right)=M_f \quad (15)$$

$$P\left(t_k^+\right)=P\left(t_k^-\right) \quad (16)$$

$$S\left(t_k^+\right)=S\left(t_k^-\right) \quad (17)$$

III. PARTICLE SWARM OPTIMIZATION

Inspired from social behavior of bird flocking or fish schooling, Particle Swarm Optimization is a meta-heuristic global optimization technique introduced by Kennedy and Eberhart [23]. It is part of the category called swarm intelligence (SI), which in turn is a subcategory of evolutionary computation (EC) [24] and it is easily implemented in most programming languages and has proven to be both very fast and effective when applied to a diverse set of optimization problem.

The system initially has a population of random selective solutions. Each potential solution is called a particle.

The particles are “flown” through the problem space by following the current optimum particles. Each particle keeps track of its coordinates in the problem space, which are associated with the best solution (fitness) that it has achieved so far. This implies that each particle has memory, which allows it to remember the best position on the feasible search space that has ever visited. This value is commonly called *pbest*. Another best value that is tracked by the particle swarm optimizer is the best value obtained so far by any particle in the neighborhood of the particle. This location is commonly called *gbest*. The basic concept behind the PSO technique consists of change in the velocity (or accelerating) of each particle toward its *pbest* and *gbest* positions at each time step [25].

The principle of PSO technique is presented in [12]:

Each particle is characterized by a position p and velocity v. During flight, each particle updates its own velocity and position by taking benefit from its best experience and the best experience of the entire population [26].

Let k be the iteration index. The new particle velocity and position are updated according to the move equations [27] [28]:

$$v_{k+1} = w_k v_k + b_1 r_1 \left(pbest_k - p_k \right) + b_2 r_2 \left(pgbest_k - p_k \right) \quad (18)$$

$$p_{k+1} = p_k + v_{k+1} \quad (19)$$

with :

- p_k : position of each particle at iteration k
- v_k : velocity of each particle at iteration k
- b_1 and b_2 : strength of attraction, fixed positive coefficients of acceleration
- r_1 and r_2 : two random numbers drawn uniformly in the interval [0,1]
- $pbest_k$: best position discovered by the particle until the iteration index k.
- $pgbest_k$: global best particle position of the entire population.

Inertia weight w controls the impact of the previous velocities on the current velocity. It influences the tradeoff between the global and local exploitation abilities of the particles. For initial stages of the search process, large inertia weight to enhance the global exploitation is recommended while for last stages, the inertia weight is reduced for better local exploration [26].

Weight is updated as:

$$w_k = w_{max} - \left(\frac{w_{max} - w_{min}}{max\_I} \right) .k \quad (20)$$

where w_{min} and w_{max} are minimum and maximum values of w and *max_I* represents the number of maximal iterations.

At each iteration, the behavior of a given particle is a compromise among three possible choices:

- to follow its own way,
- to go toward its best previous position,
- to go toward the best neighbor.

In this paper, we use the PSO to find the optimal switching instants minimizing the functional cost J. We choose a random population of particles where each particle represents the switching instants from t_1 to t_K. $K+1$ is the number of subsystems. The proposed optimization algorithm is composed of five steps as in [12], [29]

IV. Numerical Example

To illustrate the proposal approach, we consider a linear controlled switched system which was treated in many optimal control studies [1],[13], [30], [31], [32]:

- subsystem 1:

$$\dot{x} = A_1 x + B_1 u = \begin{bmatrix} 0.6 & 1.2 \\ 0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

- subsystem 2:

$$\dot{x} = A_2 x + B_2 u = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u$$

We introduce an inequality pure state constraint:

$$x_2(t) \le 2,7$$

We assume that $t_0 = 0$ and $t_f = 2$ s. The system switched at $t = t_1$ from subsystems 1 to 2, such that $0 \le t_1 \le t_f$. We want to find an optimal switching instant t_1 that minimizes the criterion :

$$J = \frac{1}{2}\left(x_1\left(t_f\right) - 4\right)^2 + \frac{1}{2}\left(x_2\left(t_f\right) - 2\right)^2 + \frac{1}{2}\int_0^{t_f}\left(\left(x_2\left(t\right) - 2\right)^2 + u^2\left(t\right)\right)dt$$

and while respecting the constraint such that:

$x_1(0) = 0$ and $x_2(0) = 2$

We construct the Lagrangian:

$$\Lambda = \frac{1}{2}\left(\left(x_2\left(t\right) - 2\right)^2 + u^2\left(t\right)\right) + \lambda\left(x_2\left(t\right) - 2,7\right)$$

By applying the KKT conditions in (6), we obtain the Lagrange multiplier $\lambda = 0,7$ and the augmented Hamiltonian is:

- for $t \in [t_0, t_1)$

$$H_1 = \frac{1}{2}\left(\left(x_2(t)-2\right)^2 + u^2\right) + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}(A_1x + B_1u) + 0,7\left(x_2(t) - 2,7\right)$$

- and for $t \in [t_1, t_f]$

$$H_2 = \frac{1}{2}\left(\left(x_2(t)-2\right)^2 + u^2\right) + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}(A_2x + B_2u) + 0,7\left(x_2(t) - 2,7\right)$$

For the PSO algorithm, we take the following parameters:
Swarm size : 40
Maximal number of iterations : 30
$b_1 = b_2 = 0,75$
$w_{max} = 0.9$ and $w_{min} = 0.4$
and obtain the optimal switching instant $t_{1opt} = 0.1880s$ for the corresponding optimal cost $J_{opt} = 9.9269$.

Figures 1, 2 and 3 show the continuous state trajectory evolution while Figure 4 presents the continuous control input evolution.

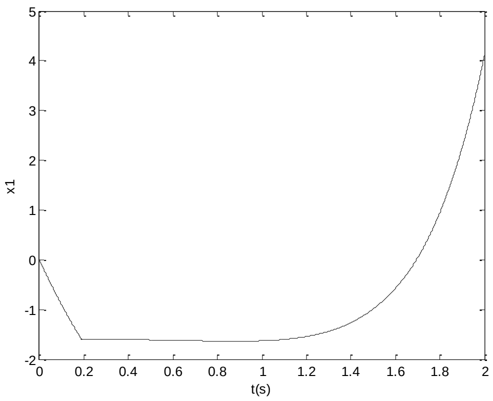

Fig. 1. x_1 evolution trajectory

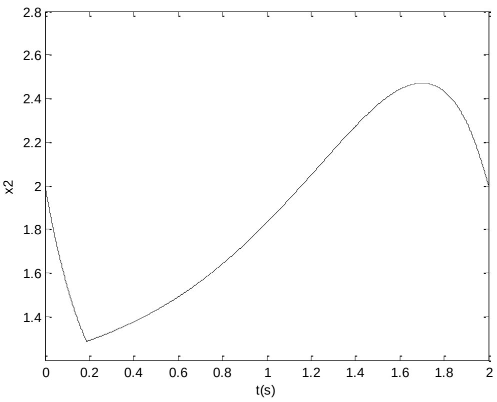

Fig. 2. x_2 evolution trajectory

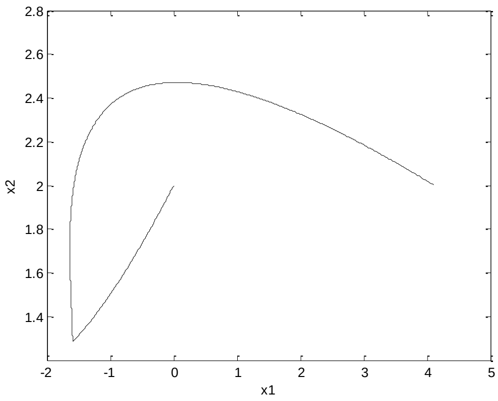

Fig. 3. The state evolution trajectory

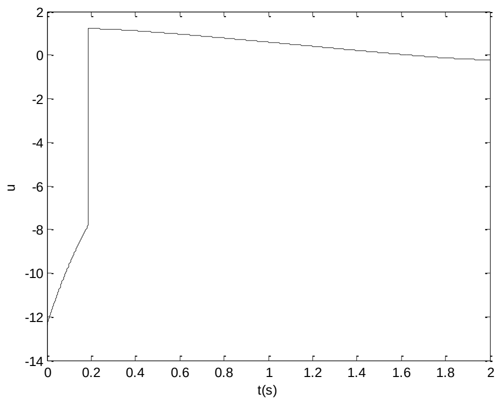

Fig. 4. The input evolution trajectory

V. CONCLUSION

The optimal control of switched systems requires the determination of the optimal inputs and switching instants, but it has to consider the state or the input constraints. This is due to the logical presence of these constraints in the real switched processes. In this paper, we have dealt with the case of linear switched systems under inequality state constraints. We have studied its optimal control using the Pontryagin principle and the KKT conditions for the input, and a PSO algorithm for the determination of optimal switching instants. By the end of this research, one can conclude that this method is very efficient.

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