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IMAGE PROCESSING BY A FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

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***Abstract*—Many fractional-order based methods have been used in image processing field, and many methods are developed to solve the problem of fractional systems. The traditional integer-order partial differential equation-based image denoising approaches often blur the edge and complex texture detail; thus, their denoising effects for texture image are not very good. To solve the problem, a fractional partial differential equation-based denoising model for texture image is proposed, which applies a novel mathematical method—fractional calculus to image processing from the view of system evolution. We know from previous studies that fractional-order calculus has some unique properties comparing to integer-order differential calculus that it can nonlinearly enhance complex texture detail during the digital image processing.**

***Index Terms*—Topological degree, elliptic problem, homotopy, image restoration**

I. Introduction

Removing noise while preserving fine details is a challenging issue in image processing. One classical partial differential equation (PDE) based technique is the total variation (TV) minimization, which was inaugurated in [1] by Rudin and al. depicted as

$$\min_{u\in BV(\Omega)\cap L^2(\Omega)} \int_\Omega |Du| + \frac{\gamma}{2}\int_\Omega |u-u_0|^2\,dx, \qquad (1)$$

where Ω denotes a bounded open domain with a Lipschitzian boundary, u and u_0 represent the original image and the observed image respectively. Furthermore, to improve the edge-preserving capability, T. CHAN and Chan [2,3] presented the adaptive TV approach to image restoration.

The PM model has good performance in flat regions with uniform intensity distribution,and the TV model works better inpreserving edges X.Zhang et al.[5] proposed anovel model (i.e., PMTV model) by weighted combinations of PM model and TV model A. Yahya et al.[6] proposed a new denoising technique by blending isotropic diffusion, PM model, and TV model. Although the above second-order PDEs can reduce noise level while preserving the image features, they tend to make the processed image look "blocky", because the images used by second-order PDEs to approximate an observed image are often piecewise constant. In order to reduce blocky effect, a class of four-order PDEs were introduced by You and Kaveh in 2000 [7], but these methods often lead to speckle effect.

To overcome those aforementioned limitations, fractionalorder PDEs have recently been researched and applied to the field of image processing and computer vision. For example, Bai and Feng [8] proposed a class of FPM models for image denoising, in which the energy function is defined as:

The main goal of this work is to apply an adaptive fractional order regularization term for the restoration of textured images corrupted by additive noise and blur. To achieve this aim, we use a 2-phase approach. First we apply a suitable texture detection method on the observed image to obtain a texture map. Then a fractional order regularization is applied to the parts of the image which are characterized to be texture regions by the map and the classical TV regularization ($\ell^1 - TV$) is applied elsewhere. In particular, we propose to replace the TV regularization term $\|u\|_{TV}$ in 1 with a spatially adaptive fractional order TV regularization term, thus integrating the following four ingredients:

- use of the fractional order α of derivatives to better preserve textures,
- spatial adaptivity of α in order to allow flexibility in choosing the correct regularizing operator,
- spatial adaptivity of γ in order to locally control the extent of restoration over image regions according to their content,
- an effective texture detection methodology based on the noise auto-correlation energy which makes no assumption about the noise level of the image. This inspired part of the work [5].

II. The proposed adaptive model

We propose to modify the functional in 1 to the following adaptive fractional variational model:

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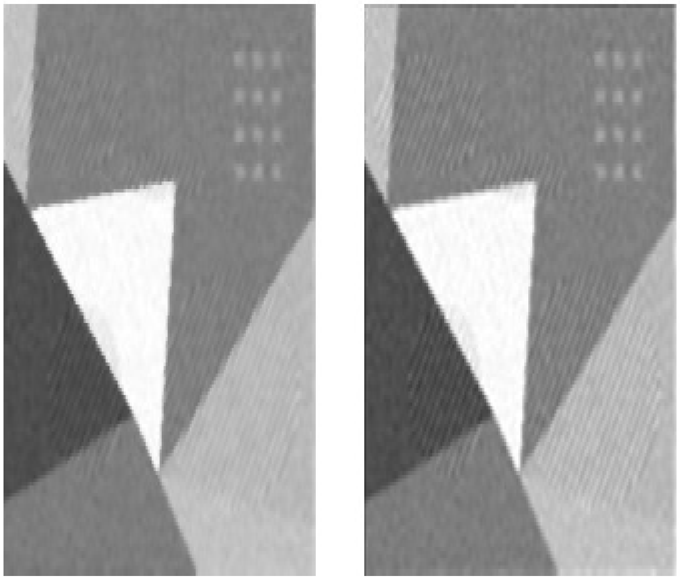

Fig. 1: $\alpha = 1.0 and 1.5$

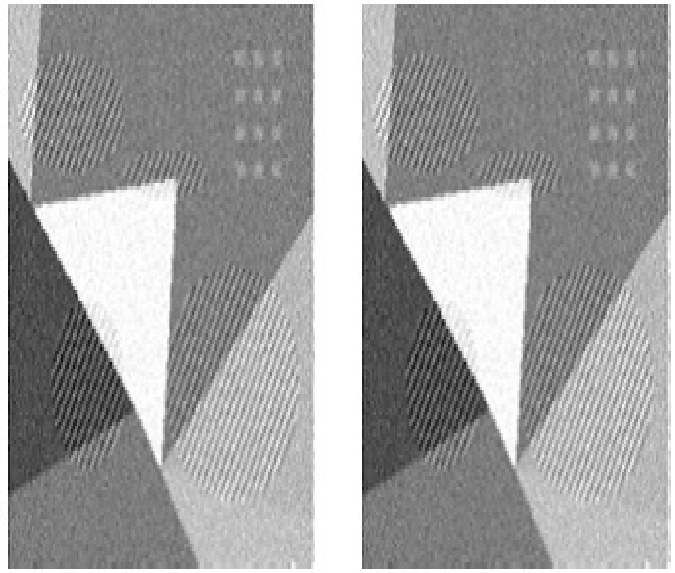

Fig. 2: $\alpha = 1.8,\ \alpha = 2$

$$\min_{u \in BV(\Omega)\cap L^2(\Omega)} \int_\Omega |D^\alpha (G_\sigma * u)| + \frac{\gamma}{2}\int_\Omega |u - u_0|\, dx, \quad (2)$$

where γ representing the regularization parameter for the ith pixel, where α represents the fractional order of differentiation for the ith pixel and G_σ is a Gaussian function of standard deviation σ, and

$$D^\alpha u = \left(D_x^\alpha u, D_y^\alpha u\right)^t$$

is the fractional-order discrete gradient operator, with components representing the x and y-directional fractional finite difference operators.

Let us motivate our model by analyzing the high-pass filtering character of the fractional order derivative operator. In Fig. 1 we show the restored images of the blurred and noisy test image in Fig. 2 and Fig.3 by applying model (1) while keeping $\gamma = 1.0$.

III. Texture detection method

Our idea is to use the auto-correlation function to detect non-whiteness in data. Inspired by [1], starting from the observed degraded image, i.e., $u(0) = u_0$, we apply a simple TV-flow with Neumann homogeneous boundary conditions

$$u^{(k+1)} = u^{(k)} + t\nabla\left(\frac{\nabla^\alpha u^{(k)}}{\left|\nabla G_\sigma * u^{(k)}\right|}\right) \quad (3)$$

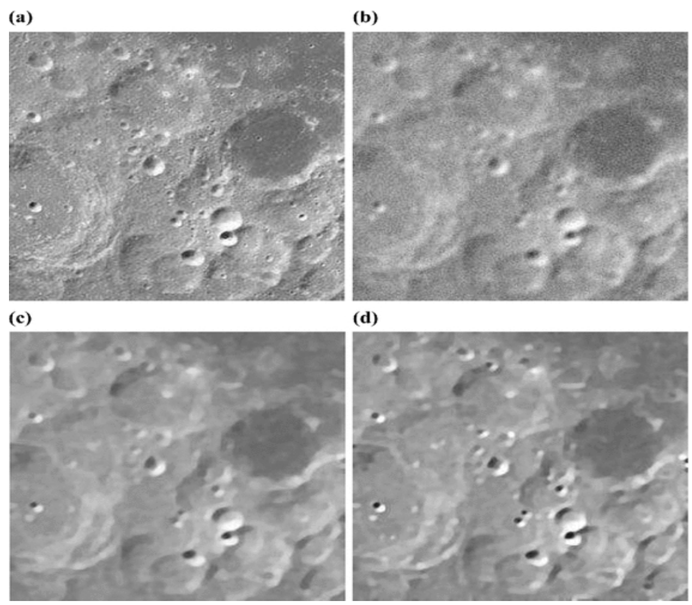

Fig. 3: (a) true image, (b) $\alpha = 1.5$ $(c)\alpha = 1.8$, $(d)\alpha = 2$

which approaches a piecewise constant image, so-called "cartoon model", that we denote by $u^{(k)}$.

Under this decomposition, the residual can be represented as

$$r^{(k)} = f - u^{(k)} = u_{nc} + e,$$

we propose to consider the auto-correlation of the residue $r^{(k)}$ and to choose $\tilde{k}$ accordingly.

To describe the details of the approach, we briefly introduce some required statistical concepts. Let $\epsilon = \{E_{i,j} : i, j = 1,\cdots,n\}$ be an $n \times n$ discrete random field with $E_{i,j}$ denoting the scalar random variable modeling noise at pixel (i, j). The auto-correlation of ϵ is a function

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ρ_ϵ mapping pairs of pixel locations (i_1, j_1), (i_2, j_2) into a scalar value that must lie in the range $[-1, 1]$, which represents the Pearson's correlation coefficient between the two corresponding random variables E_{i_1,j_1}, E_{i_2,j_2}, i.e.,

$$\rho_\epsilon [i_1, j_1, i_2, j_2] = \frac{E\left[\left(E_{i_1,j_1} - \mu_{i_1,j_1}\right)\left(E_{i_2,j_2} - \mu_{i_2,j_2}\right)\right]}{\sigma_{i_1,j_1}\sigma_{i_2,j_2}} \tag{4}$$

where E is the expected value operator, $\mu_{i,j}$ and $\sigma_{i,j}$ are the mean and standard deviation of the random variable $E_{i,j}$.

Since we assume that noise is white, i.e., wide-sense stationary, zero-mean, uncorrelated, the auto-correlation of E depends only on the lag between the two pixel locations $[l, m] = (i_2 - i_1, j_2 - j_1)$ and (4) can be rewritten as follows

$$\rho_\epsilon [l, m] = \frac{1}{\sigma^2} E\left[E_{i,j} E_{i+l,j+m}\right] = \begin{cases} 1 & if \quad (l,m) = (0,0) \\ 0 & otherwise, \end{cases} \quad l, m = 0, \cdots, n-1 \tag{5}$$

independently on i, j. That is, a white noise is characterized by zero values of the autocorrelation function at all non-vanishing lags.

Moreover, assuming that the noise process is also ergodic, provided that the observed realization e of the noise random field ϵ is "sufficiently long", implies that ρ_ϵ in (5) is well estimated by the sample auto-correlation function of e defined as

$$\widehat{\rho}_\epsilon [l, m] = \frac{1}{n^2 \widehat{\sigma}^2} \sum_{i,j=1}^{n} e_{i,j} e_{i+l,j+m}, \tag{6}$$

where $\widehat{\sigma}^2$ is the sample variance of the observed noise realization e. We remark that, for a generic observed realization x, the sample auto-correlation $\widehat{\rho}_x[l, m] \in [-1, 1]$, with 1 indicating perfect correlation and -1 indicating perfect anti-correlation.

In order to find a characteristic scale $\tilde{k}$ to detect textures, we propose to minimize the following residual auto-correlation energy

$$J_{r^{(k)}} = \max_{[l,m] \neq [0,0]} \left|\widehat{\rho_{r^{(k)}}} [l, m]\right|, \tag{7}$$

that, according to 6, for a cartoon image corrupted by white noise should be zero. For a cartoon image without textures, the energy $J_{r^{(k)}}$ monotonically decreases and vanishes. In the presence of textures, initially, the TV-flow makes the residual image be essentially given by noise, so that the auto-correlation energy $J_{r^{(k)}}$ decreases. As soon as the texture part u_{nc} initiates to contaminate the residual, the energy $J_{r^{(k)}}$ starts increasing since textures are typically correlated.

Our proposal is based on the idea to find the characteristic scale $\tilde{k}$ which makes the auto-correlation energy of the residual image $J_{r^{(k)}}$ minimal.

IV. The numerical algorithm

The fidelity and regularization terms in 1 are not differentiable, therefore in the following we use a smoothed version of them. To this end, let us define $|v|_\varepsilon = \sqrt{v^2 + \varepsilon}$ for any $v \in \mathbb{R}$ and $\varepsilon > 0$, let β and γ be two small regularization parameters. Hence, we want to minimize the functional

Algorithm 1.1: Texture Detetion (TD) Algorithm

Input: degraded image u_0 , number of texture classes C ;

Output: texture-adaptive parameters γ_i , α_i , $i = 1, \cdots, n^2$;

1) Initialize the iterative process by setting $u(0) = u_0$;
2) Repeat
3) perform one step of the TV flow $u^{(k+1)} = TV(u^{(k)})$ by 3
4) compute the residue image $r^{(k+1)} = u_0 - u^{(k+1)}$
5) compute the residue auto-orrelation $3c1_{r^{(k+1)}}$ by 6
6) compute the residue auto-orrelation energy $J_{r^{(k+1)}}$ by 7
7) until $J_{r^{(k+1)}} > J_{r^{(k)}}$
8) $\tilde{k} := k$ harateristi sale found at the rst lo al minimum;
9) $T =$ ComputeTexture Measure $(u^{(\tilde{k})})$ with T taking values in $[0, 1]$;
10) partition T into C classes $T_1, T_2, \cdots, T_C$;
11) assign (λ_i, α_i) a ording to T_i for $i = 1, \cdots, C$.

V. CONCLUSION

This work proposes new applications of fractional-order partial differential equations in image processing. Our studies led to proposing a general reconstruction algorithm that incorporates the fractional derivative implementation from [1]. Concerning denoising, better results are obtained with an order α which is fractional rather than integer. The interesting values for the fractional order α seem to be around 1.5 and 1.75. It corroborates previous results [1]. Contrary to existing iterative processes with a fractional order, the algorithm presented here is non iterative. It gives similar results for a shorter computer time and can be used to solve texture problems. The comparison with state-of-the-art methods involving partial differential equations showed better results in terms of quality.

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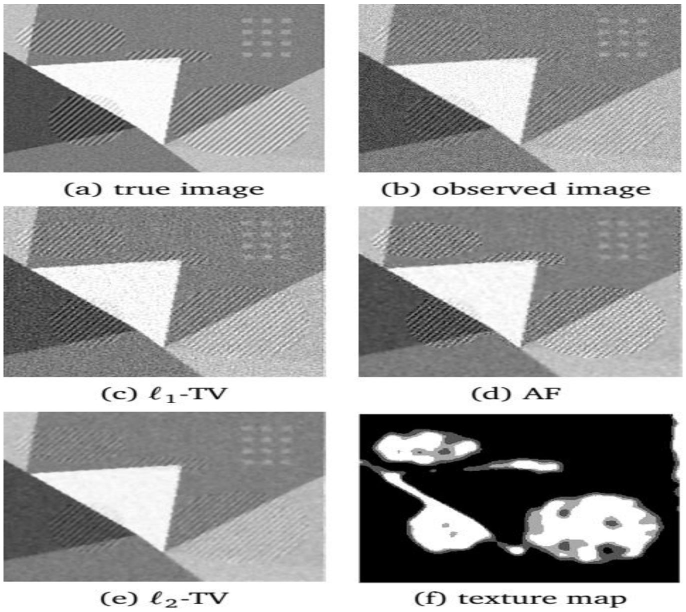

Fig. 4: (a) blur- and noise-free image;
(b) the corrupted image produced by Gaussian blur, defined by the parameters $band = 3$ and $sigma = 1.5$ and by 0.1 of Gaussian noise;
(c) restoration with ℓ^1 -TV with $\lambda_1 = 0.1$;
(d) restored image determined by AF algorithm;
(e) restoration with ℓ^2- TV with $\lambda_1 = 0.1$;
(f) texture classes computed on (b)

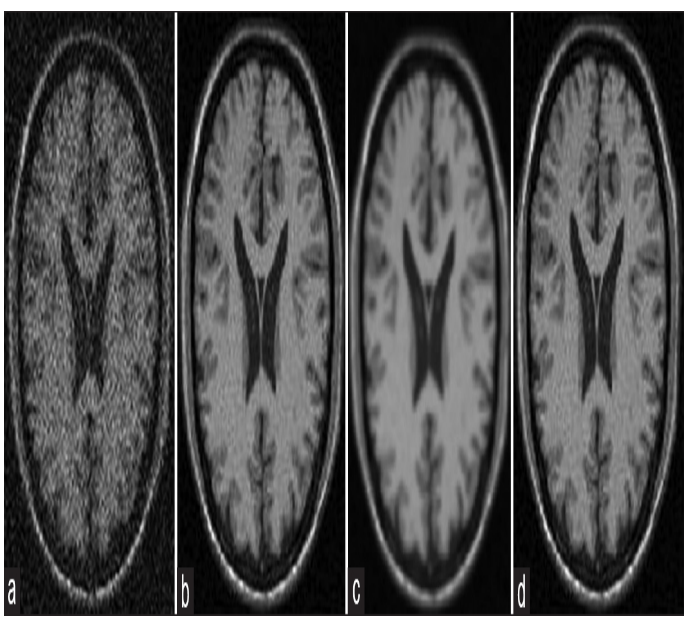

Fig. 5: (a) blur- and noise-free image;
(b) the corrupted image produced by Gaussian blur, defined by the parameters $band = 3$ and $sigma = 1.5$ and by 0.1 of Gaussian noise;
(c) restoration with ℓ^1 -TV with $\lambda_1 = 0.1$;
(d) restored image by our model;

Iteration	$PSNR_{OM}$	$PSNR_{TV}$	$PSNR_{PM}$
1	20.8668	20.8529	20.8520
100	21.2799	21.1378	21.1190
200	21.5155	21.3650	21.3607
300	21.6672	21.5327	21.5300
400	21.7134	21.6251	21.6340
500	21.7720	21.7184	21.7105
600	21.7989	21.7502	21.6962

TABLE I: PSNR comparisons for the three models (for image 01)

Iteration	$PSNR_{OM}$	$PSNR_{TV}$	$PSNR_{PM}$
1	20.8694	20.8591	20.8381
100	21.2846	21.1174	21.0856
200	21.5200	21.3370	21.2816
300	21.6341	21.4951	21.4376
400	21.6781	21.6149	21.5518
500	21.7456	21.6787	21.6234
600	21.7628	21.7153	/

TABLE II: PSNR comparisons for the three models (for image 02)

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